A Fourier Spectral Method for the Rosenau Equation

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Introduction

In this study we consider the **Rosenau equation** with power type nonlinearity:

$$u_t + u_x + u_{xxxxt} + (u^p)_x = 0. \tag{1}$$

Rosenau equation is derived to model dense discrete dynamical systems [1]. The global well-posedness of Cauchy problem associated with (1) studied in [2] for the initial data $u(x,0) \in H_0^4(\mathbb{R})$.

We propose a numerical scheme combining Fourier pseudo-spectral method in *space* and a second order finite difference method in *time*. We prove that our numerical scheme has accuracy of order $O(\Delta t^2 + h^m)$.

We then present several numerical experiments to check the efficiency of our numerical scheme. For this aim, we first generate an *initial* solitary wave solution by using Petviashvili's method, as there are *no exact* solitary wave solutions of (1). We investigate the time evolution of numerically generated solution by using the proposed scheme given by (2). The errors in conserved quantities are also presented.

Conserved Quantities

A **Lagrangian** for the Rosenau equation is given by:

$$\mathcal{L}(U_x, U_t, U_{xxx}, U_{xxt}) = U_x U_t + U_{xxx} U_{xxt} + (U_x)^2 + \frac{2}{p+1}(U_x)^{p+1}$$

where $u := U_x$ and $p \in \mathbb{Z}_{\geq 0}$ [3]. The following conserved quantities are obtained by using the **Noether's theorem**.

Energy	$\int_{\mathbb{R}} (u^2 + \frac{2}{p+1} u^{p+1}) dx$
Momentum	$\int_{\mathbb{R}} (u^2 + u_{xx}^2) dx$
Mass	$\int_{\mathbb{R}} u dx$

Discretization

We discretize Rosenau equation (1) as follows:

$$(1 + D_N^4)\left(\frac{U^{n+1} - U^{n-1}}{2\Delta t}\right) + D_N\left(\frac{U^{n+1} + U^{n-1}}{2}\right) + D_N((U^n)^p) = 0 \tag{2}$$

and for finding U^1, we use the following:

$$(1 + D_N^4)\left(\frac{U^1 - U^0}{\Delta t}\right) + D_N\left(\frac{U^1 + U^0}{2}\right) + D_N((U^0)^p) = 0.$$

We solve (2) by using **FFT algorithm**.

Lemma

Let f be *any* element of $H^3(0,T)$ and $\frac{T}{\Delta t} \in \mathbb{Z}_{\geq 2}$, φ be *any* element of $\mathcal{B}^N$, $m \in \mathbb{Z}_{\geq 1}$, $\Omega := (0,L)$ and $g \in L^2(\Omega)$. Then we have following estimates:

$$\sqrt{\Delta t \sum_{k=1}^{\frac{T}{\Delta t}-1} \left| \frac{f^{k+1} - f^{k-1}}{2\Delta t} - \partial_t f^k \right|^2} \leq C\Delta t^2 ||f||_{H^3(0,T)} \tag{3}$$

$$\sqrt{\Delta t \sum_{k=1}^{\frac{T}{\Delta t}-1} \left| \frac{f^{k+1} - 2f^k + f^{k-1}}{\Delta t^2} \right|^2} \leq C||f||_{H^2(0,T)} \tag{4}$$

$$||\mathcal{I}_N \varphi||_{H^1} \leq ||\varphi||_{H^1} \tag{5}$$

$$||g(x) - \mathcal{I}_N g(x)||_{L^2(\Omega)} \leq Ch^m ||g(x)||_{H^m(\Omega)} \tag{6}$$

Truncation Error Analysis in Time

Assume that solution u of (1) belongs to $L^\infty(0,T;H^{m+1}) \cap H^3(0,T;H^4)$. Let $U_N(x,t) := \mathcal{P}_N u(x,t)$ and $U^n := \mathcal{I} U_N$ be its discrete interpolation. Then

$$(1 + D_N^4)\left(\frac{U^{n+1} - U^{n-1}}{2\Delta t}\right) + D_N\left(\frac{U^{n+1} + U^{n-1}}{2}\right) + D_N((U^n)^p) = \tau^n$$

where τ^n satisfies the estimate

$$||\tau||_{\ell^2(0,T;\ell^2)} \leq C(\Delta t^2 + h^m)$$

and C *only depends* on exact solution u.

Sketch of Proof : Analysis of linear terms follows from estimates (3) and (4). Also, we can estimate nonlinear term $(U^n)^p$ by using (5) and (6).

Solitary Wave Solutions

The ansatz $u(x,t) = Q(x - ct)$ gives $(c-1)Q + cQ^{(4)} - Q^p = 0$.

Non-existence results can be derived from the Pohozaev type identities [4].

- $c < 0$ and p is odd,
- $0 < c < 1$ for all $p > 1$.

Existence result: Theorem 2.1 of [5] states the existence of Q for all $c, p > 1$.

Petviashvili Method

Applying the Fourier transform gives: $(c - 1 + ck^4)\widehat{Q}(k) - \widehat{Q^p}(k) = 0$.
The method is:

$$\widehat{Q}_{n+1}(k) = \frac{M_n^\gamma}{c - 1 + ck^4} \widehat{Q_n^p}(k)$$

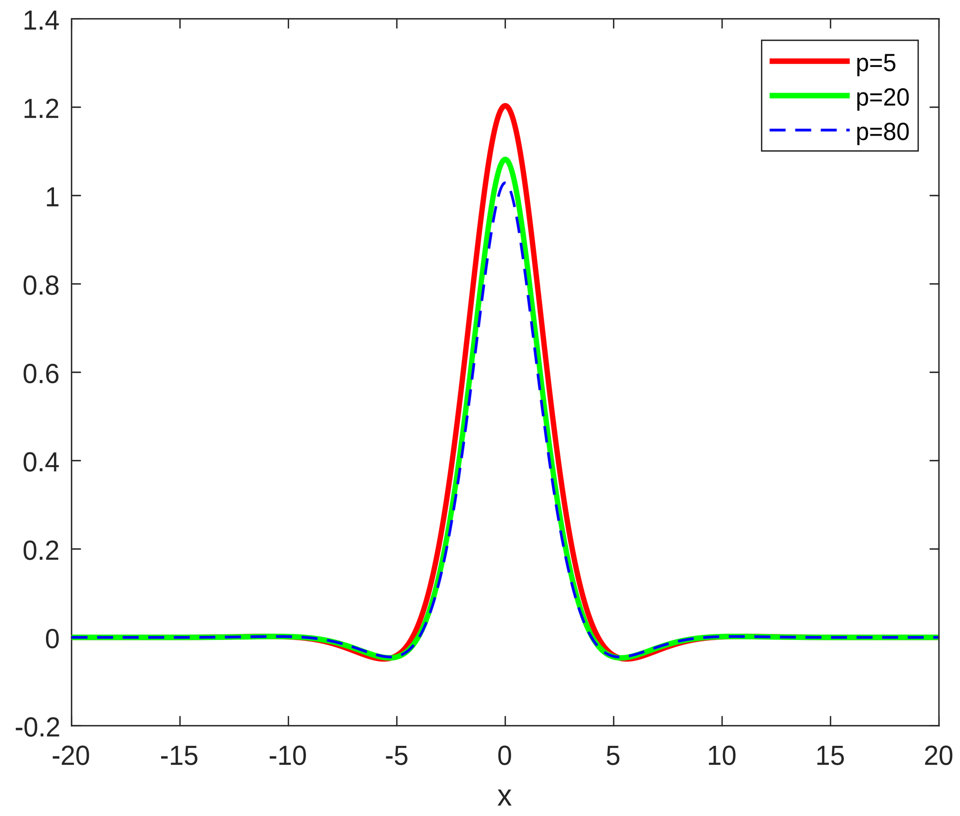
with stabilizing factor

$$M_n = \frac{\int_{\mathbb{R}} (c - 1 + ck^4)(\widehat{Q}_n(k))^2 dk}{\int_{\mathbb{R}} \widehat{Q}_n(k) \widehat{Q_n^p}(k) dk}$$

and optimal value of γ is $p/(p-1)$. See [6] for the convergence properties of this method.

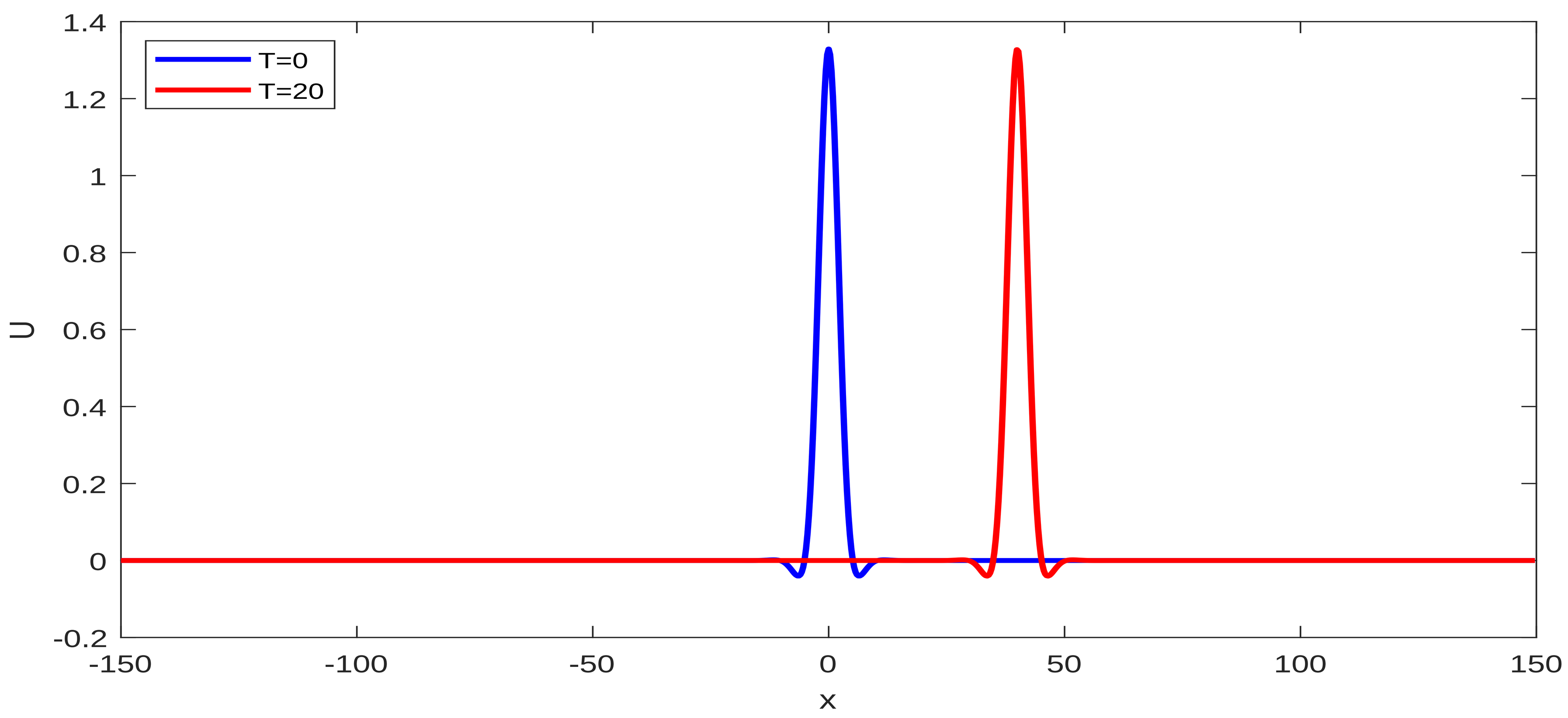
Wave Profiles

Solitary wave profiles for various nonlinearities with $c = 2$.



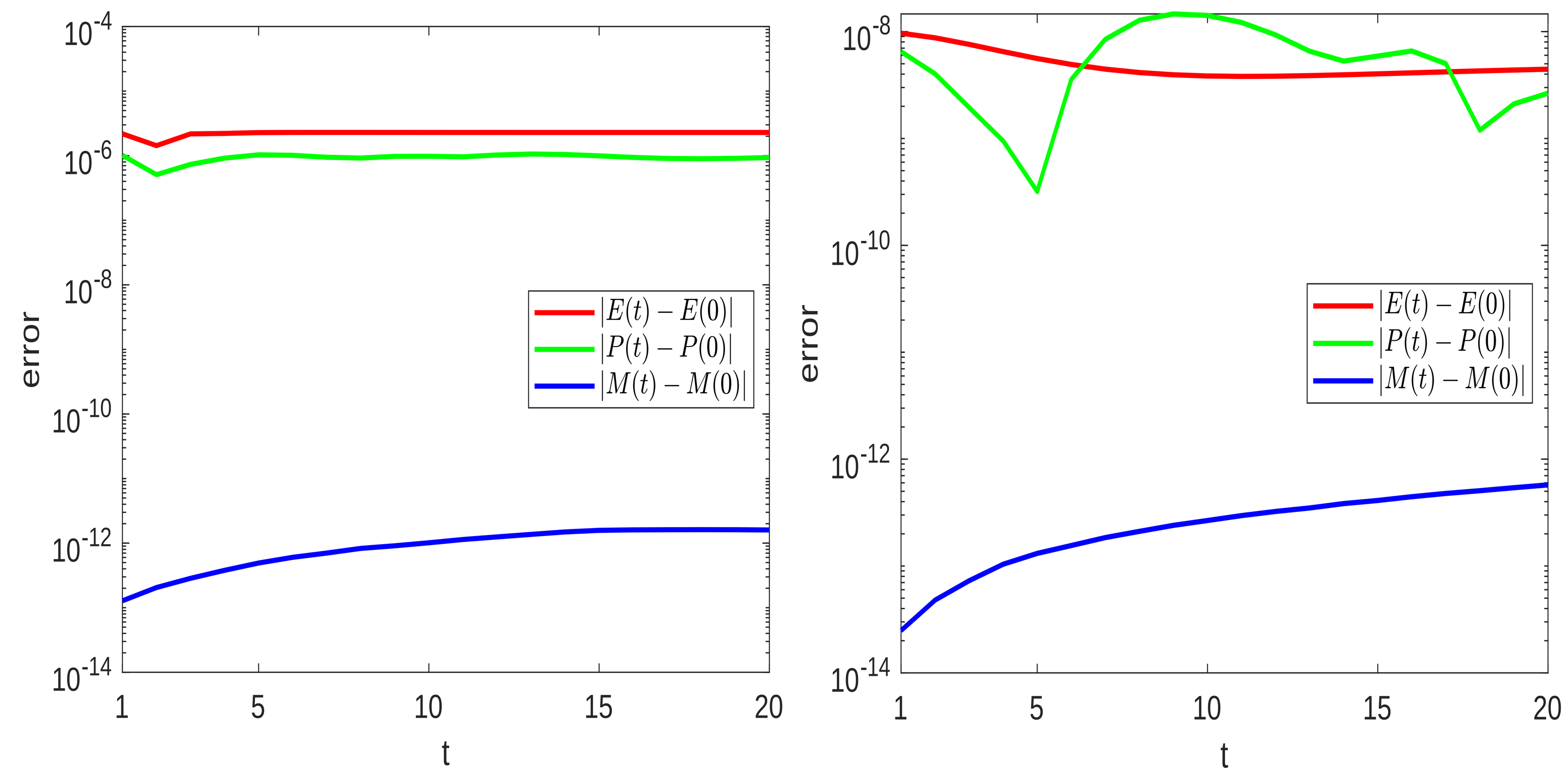
Time Evolution of Initial Solitary Wave

Time evolution of solitary wave with $c = 2$ and $p = 2$. Here the space and time grid numbers are $N = 1000$ and $M = 20000$ respectively.



Relative Errors

Relative errors of energy E, momentum P and mass M for $c = 2$ (left panel) and $c = 1.2$ (right panel).



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